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## Kingman, category and combinatorics

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## Abstract

Kingman's Theorem on skeleton limits—passing from limits as  $n \rightarrow \infty$  along  $nh$  ( $n \in \mathbb{N}$ ) for enough  $h > 0$  to limits as  $t \rightarrow \infty$  for  $t \in \mathbb{R}$ —is generalized to a Baire/measurable setting via a topological approach. We explore its affinity with a combinatorial theorem due to Kestelman and to Borwein and Ditor, and another due to Bergelson, Hindman and Weiss. As applications, a theory of 'rational' skeletons akin to Kingman's integer skeletons, and more appropriate to a measurable setting, is developed, and two combinatorial results in the spirit of van der Waerden's celebrated theorem on arithmetic progressions are given.

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## 1 Introduction

The background to the theme of the title is Feller's theory of *recurrent events*. This goes back to Feller in 1949 [F1], and received its first

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textbook synthesis in [F2] (see e.g. [GS] for a recent treatment). One is interested in something ('it', let us say for now—we can proceed informally here, referring to the above for details) that happens (by default, or by fiat) at time 0, may or may not happen at discrete times  $n = 1, 2, \dots$ , and is such that its happening 'resets the clock', so that if one treats this random time as a new time-origin, the subsequent history is a probabilistic replica of the original situation. Motivating examples include return to the origin in a simple random walk (or coin-tossing game); attaining a new maximum in a simple random walk; returning to the initial state  $i$  in a (discrete-time) Markov chain. Writing  $u_n$  for the probability that 'it' happens at time  $n$  (so  $u_0 = 1$ ), one calls  $u = (u_n)$  a *renewal sequence*. Writing  $f_n$  for the probability that 'it' happens *for the first time* at  $n > 1$  ( $f_0 := 0$ ),  $f = (f_n)$ , the generating functions  $U, F$  of  $u, f$  satisfy the *Feller relation*  $U(s) = 1/(1 - F(s))$ .

It is always worth a moment when teaching stochastic processes to ask the class whether time is discrete or continuous. It is both, but which aspect is uppermost depends on how we measure, or experience, time—whether our watch is digital or has a sweep second hand, one might say. In continuous time, one encounters analogues of Feller's theory above in various probabilistic contexts—e.g., the server in an  $M/G/1$  queue being idle. In the early 1960s, the Feller theory, queueing theory (in the phase triggered by Kendall's work, [Ken1]) and John Kingman were all young. Kingman found himself drawn to the task of creating a continuous-time version of Feller's theory of recurrent events (see [King4] for his reminiscences of this time)—a task he triumphantly accomplished in his theory of *regenerative phenomena*, for which his book [King3] remains the standard source. Here the role of the renewal sequence is played by the Kingman  $p$ -function, where  $p(t)$  is the probability that the regenerative phenomenon  $\Phi$  occurs at time  $t \geq 0$ .

A continuous-time theory contains within itself infinitely many versions of a discrete-time theory. For each fixed  $h > 0$ , one obtains from a Kingman regenerative phenomenon  $\Phi$  with  $p$ -function  $p(t)$  a Feller recurrent event (or regenerative phenomenon in discrete time, as one would say nowadays),  $\Phi_h$  say, with renewal sequence  $u_n(h) = p(nh)$ —called the discrete *skeleton* of  $\Phi$  for time-step  $h$ —the  $h$ -skeleton, say.

While one can pass from continuous to discrete time by taking skeletons, it is less clear how to proceed in the opposite direction—how to combine discrete-time information for various time-steps  $h$  to obtain continuous-time information. A wealth of information was available in the discrete-time case—for example, limit theorems for Markov chain

transition probabilities. It was tempting to seek to use such information to study corresponding questions in continuous time, as was done in [King1]. There, Kingman made novel use of the Baire category theorem, to extend a result of Croft [Cro], making use of a lemma attributed both to Golomb and Gould and to Anderson and Fine (see [NW] for both).

While in the above we have limits at infinity through integer multiples  $nx$ , we shall also be concerned with limits through positive rational multiples  $qx$  (we shall always use the notations  $\lim_n$  and  $\lim_q$  for these). There are at least three settings in which such rational limits are probabilistically relevant:

- (i) *Infinitely divisible  $p$ -functions*. The Kingman  $p$ -functions form a semigroup under pointwise multiplication (if  $p_i$  come from  $\Phi_i$  with  $\Phi_1, \Phi_2$  independent,  $p := p_1 p_2$  comes from  $\Phi := \Phi_1 \cap \Phi_2$ , in an obvious notation). The arithmetic of this semigroup has been studied in detail by Kendall [Ken2].
- (ii) *Embeddability of infinitely divisible laws*. If a probability law is infinitely divisible, one can define its  $q$ th convolution power for any positive rational  $q$ . The question then arises as to whether one can embed these rational powers into a continuous semigroup of real powers. This is the question of *embeddability*, studied at length (see e.g. [Hey, Ch. III]) in connection with the Lévy–Khintchine formula on locally compact groups.
- (iii) *Embeddability of branching processes*. While for simple branching processes both space (individuals) and time (generations) are discrete, it makes sense in considering e.g. the biomass of large populations to work with branching processes where space and/or time may be continuous. While one usually goes from the discrete to the continuous setting by taking limits, embedding is sometimes possible; see e.g. Karlin and McGregor [KMcG], Bingham [Bin].

In addition, (i) led Kendall [Ken2, Th. 16] to study *sequential regular variation* (see e.g. [BGT, §1.9]). The interplay between the continuous and sequential aspects of regular variation, and between measurable and Baire aspects, led us to our recent theory of *topological regular variation* (see e.g. [BOst2] and our other recent papers), our motivation here.

In Section 2 we discuss the relation between Kingman’s Theorem and the Kestelman–Borwein–Ditor Theorem (KBD), introducing definitions and summarizing background results which we need (including the density topology). In Section 3 we generalize to a Baire/measurable setting the Kingman Theorem (originally stated for open sets). Our

(bi-)topological approach (borrowed from [BOst7]) allows the two cases to be treated as one that, by specialization, yields either case. (This is facilitated by the density topology.) The theorem is applied in Section 4 to establish a theory of ‘rational’ skeletons parallel to Kingman’s integer skeletons. In Section 5 we offer a new proof of KBD in a ‘consecutive’ format suited to proving in Section 6 combinatorial results in the spirit of van der Waerden’s celebrated theorem on arithmetic progressions. Again a bitopological (actually ‘bi-metric’) approach allows unification of the Baire/measurable cases. Our work in Section 6 is based on a close reading of [BHW], our debt to which is clear.

## 2 Preliminaries

In this section we motivate and define notions of contiguity. Then we gather classical results from topology and measure theory (complete metrizability and the density topology). We begin by recalling the following result, in which the expression ‘for generically all  $t$ ’ means for all  $t$  except in a meagre or null set according to context. We use the terms Baire set/function to mean a set/function with the Baire property. Evidently, the interesting cases are with  $T$  Baire non-meagre/measurable non-null. The result in this form is due in the measure case to Borwein and Ditor [BoDi], but was already known much earlier albeit in somewhat weaker form by Kestelman [Kes, Th. 3], and rediscovered by Kemperman [Kem] and later by Trautner [Trau] (see [BGT, p. xix and footnote p. 10]). We note a cognate result in [HJ, Th. 2.3.7].

**Theorem KBD (Kestelman–Borwein–Ditor Theorem, KBD)** *Let  $\{z_n\} \rightarrow 0$  be a null sequence of reals. If  $T$  is Baire/Lebesgue-measurable, then for generically all  $t \in T$  there is an infinite set  $\mathbb{M}_t$  such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

We give a new unified proof of the measure and Baire cases in Section 5, based on ‘almost-complete metrizability’ (= *almost complete* + *complete metrizability*, see below) and a *Generic Dichotomy* given in Section 3; earlier unification was achieved through a bitopological approach (as here to the Kingman Theorem) in [BOst7]. This result is a theorem about additive infinite combinatorics. It is of fundamental and unifying importance in contexts where additive structure is key; its varied applications include proofs of classical results such as Ostrowski’s Theorem on

the continuity of Baire/Lebesgue convex (and so additive) functions (cf. [BOst8]), a plethora of results in the theory of subadditive functions (cf. [BOst1, BOst4]), the Steinhaus Theorem on distances [BOst8, BOst6] and the Uniform Convergence Theorem of regular variation [BOst3]. Its generalizations to normed groups may be used to prove the Uniform Boundedness Theorem (see [Ost]). Recently it has found applications to additive combinatorics in the area of Ramsey Theory (for which see [GRS, HS]), best visualized in the language of colour: one seeks monochromatic structures in finitely coloured situations. Two examples are included in Section 6.

The KBD theorem is about shift-embedding of subsequences of a null sequence  $\{z_n\}$  into a *single* set  $T$  with an assumption of *regularity* (Baire/measurable). Our generalizations in Section 3 of a theorem of Kingman's have been motivated by the wish to establish 'multiple embedding' versions of KBD: we seek conditions on a sequence  $\{z_n\}$  and a *family* of sets  $\{T_k\}_{k \in \omega}$  which together guarantee that *one* shift embeds (different) subsequences of  $\{z_n\}$  into *all* members of the family.

Evidently, if  $t + z_n$  lies in several sets infinitely often, then the sets in question have a common limit point, a sense in which they are contiguous at  $t$ . Thus *contiguity conditions* are one goal, the other two being *regularity conditions* on the family, and *admissibility conditions* on the null sequences.

We view the original Kingman Theorem as studying contiguity at infinity, so that divergent sequences  $z_n$  (i.e. with  $z_n \rightarrow +\infty$ ) there replace the null sequences of KBD. The theorem uses *openness* as a regularity condition on the family, *cofinality* at infinity (e.g. *unboundedness* on the right) as the simplest contiguity condition at infinity, and

$$\frac{z_{n+1}}{z_n} \rightarrow 1 \text{ (multiplicative form), } z_{n+1} - z_n \rightarrow 0 \text{ (additive form) } (*)$$

as the admissibility condition on the divergent sequence  $z_n$  ((\*) follows from regular variation by Weissman's Lemma, [BGT, Lemma 1.9.6]). Taken together, these three guarantee multiple embedding (at infinity).

One can switch from  $\pm\infty$  to 0 by an inversion  $x \rightarrow 1/x$ , and thence to any  $\tau$  by a shift  $y \rightarrow y + \tau$ . *Openness* remains the *regularity* condition, a property of *density at zero* becomes the analogous *admissibility* condition on null sequences, and cofinality (or accumulation) at  $\tau$  the contiguity condition. The transformed theorem then asserts that for admissible null sequences  $\zeta_n$  there exists a scalar  $\sigma$  such that the sequence

$\sigma\zeta_n + \tau$  has subsequences in all the open sets  $T_k$  provided these all accumulate at  $\tau$ .

In the next section, we will replace Kingman's regularity condition of openness by the Baire property, or alternatively measurability, to obtain two versions of Kingman's theorem—one for measure and one for category. We develop the regularity theme bitopologically, working with two topologies, so as to deduce the measure case from the Baire case by switching from the Euclidean to the density topology.

**Definitions and notation (Essential contiguity conditions)** We use the notation  $B_r(x) := \{y : |x - y| < r\}$  and  $\omega := \{0, 1, 2, \dots\}$ . Likewise for  $a \in A \subseteq \mathbb{R}$  and metric  $\rho = \rho_A$  on  $A$ ,  $B_r^\rho(a) := \{y \in A : \rho(a, y) < r\}$  and  $\text{cl}_A$  denotes closure in  $A$ . For  $S$  given, put  $S^{>m} = S \setminus B_m(0)$ .  $\mathbb{R}_+$  denotes the (strictly) positive reals. When we regard  $\mathbb{R}_+$  as a multiplicative group, we write

$$A \cdot B := \{ab : a \in A, b \in B\}, \quad A^{-1} := \{a^{-1} : a \in A\},$$

for  $A, B$  subsets of  $\mathbb{R}_+$ .

Call a Baire set  $S$  *essentially unbounded* if for each  $m \in \mathbb{N}$  the set  $S^{>m}$  is non-meagre. This may be interpreted in the sense of the metric (Euclidean) topology, or as we see later in the measure sense by recourse to the density topology. To distinguish the two, we will qualify the term by referring to the category/metric or the measure sense.

Say that a set  $S \subset \mathbb{R}_+$  *accumulates essentially* at 0 if  $S^{-1}$  is essentially unbounded. (In [BHW] such sets are called *measurably/Baire large* at 0.) Say that  $S \subset \mathbb{R}_+$  *accumulates essentially* at  $t$  if  $(S - t) \cap \mathbb{R}_+$  accumulates essentially at 0.

We turn now to some topological notions. Recall (see e.g. [Eng, 4.3.23 and 24]) that a metric space  $A$  is *completely metrizable* iff it is a  $\mathcal{G}_\delta$  subset of its completion (i.e.  $A = \bigcap_{n \in \omega} G_n$  with each  $G_n$  open in the completion of  $A$ ), in which case it has an equivalent metric under which it is complete. Thus a  $\mathcal{G}_\delta$  subset  $A$  of the line has a metric  $\rho = \rho_A$ , equivalent to the Euclidean metric, under which it is complete. (So for each  $a \in A$  and  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $B_\delta(a) \subseteq B_\varepsilon^\rho(a)$ , which enables the construction of sequences with  $\rho$ -limit guaranteed to be in  $A$ .)

This motivates the definition below, which allows us to capture a feature of measure-category duality: both exhibit  $\mathcal{G}_\delta$  *inner-regularity*, modulo sets which we are prepared to neglect. (The definition here takes

advantage of the fact that  $\mathbb{R}$  is complete; for the general metric group context see [BOst6, Section 5].)

**Definition** Call  $A \subset \mathbb{R}$  *almost complete* (in category/measure) if

- (i) there is a meagre set  $N$  such that  $A \setminus N$  is a  $\mathcal{G}_\delta$ , or
- (ii) for each  $\varepsilon > 0$  there is a measurable set  $N$  with  $|N| < \varepsilon$  and  $A \setminus N$  a  $\mathcal{G}_\delta$ .

Thus  $A$  almost complete is Baire resp. measurable. A bounded non-null measurable subset  $A$  is almost complete: for each  $\varepsilon > 0$  there is a compact (so  $\mathcal{G}_\delta$ ) subset  $K$  with  $|A \setminus K| < \varepsilon$ , so we may take  $N = A \setminus K$ . Likewise a Baire non-meagre set is almost complete—this is in effect a restatement of Baire’s Theorem:

**Theorem B (Baire’s Theorem—almost completeness of Baire sets)** *For  $A \subseteq \mathbb{R}$  Baire non-meagre there is a meagre set  $M$  such that  $A \setminus M$  is completely metrizable.*

*Proof* For  $A \subseteq \mathbb{R}$  Baire non-meagre we have  $A = (U \setminus M_0) \cup M_1$  with  $M_i$  meagre and  $U$  a non-empty open set. Now  $M_0 = \bigcup_{n \in \omega} N_n$  with  $N_n$  nowhere dense; the closure  $F_n := \bar{N}_n$  is also nowhere dense (and the complement  $E_n = \mathbb{R} \setminus F_n$  is dense, open). The set  $M'_0 = \bigcup_{n \in \omega} F_n$  is also meagre, so  $A_0 := U \setminus M'_0 = \bigcap_{n \in \omega} U \cap E_n \subseteq A$ . Taking  $G_n := U \cap E_n$ , we see that  $A_0$  is completely metrizable.  $\square$

The tool whereby we interpret measurable functions as Baire functions is *refinement* of the usual metric (Euclidean) topology of the line  $\mathbb{R}$  to a non-metric one: the *density topology* (see e.g. [Kech, LMZ, CLO]). Recall that for  $T$  measurable,  $t$  is a (metric) density point of  $T$  if  $\lim_{\delta \rightarrow 0} |T \cap I_\delta(t)|/\delta = 1$ , where  $I_\delta(t) = (t - \delta/2, t + \delta/2)$ . By the Lebesgue Density Theorem almost all points of  $T$  are density points ([Hal, Section 61], [Oxt, Th. 3.20], or [Goff]). A set  $U$  is *d*-open (density-open = open in the density topology  $d$ ) if (it is measurable and) each of its points is a density point of  $U$ . We mention five properties:

- (i) The density topology is finer than (contains) the Euclidean topology [Kech, 17.47(ii)].
- (ii) A set is Baire in the density topology iff it is (Lebesgue) measurable [Kech, 17.47(iv)].
- (iii) A Baire set is meagre in the density topology iff it is null [Kech, 17.47(iii)]. So (since a countable union of null sets is null), the conclusion of the Baire theorem holds for the line under  $d$ .

- (iv)  $(\mathbb{R}, d)$  is a *Baire* space, i.e. the conclusion of the Baire theorem holds (cf. [Eng, 3.9]).
- (v) A function is  $d$ -continuous iff it is approximately continuous in Denjoy's sense ([Den]; [LMZ, pp. 1, 149]).

The reader unfamiliar with the density topology may find it helpful to recall Littlewood's Three Principles ([Lit, Ch. 4], [Roy, Section 3.6, p. 72]): general situations are 'nearly' the easy situations—i.e. are easy situations modulo small sets. Theorem 3.0 below is in this spirit. We refer now to Littlewood's Second Principle, of a measurable function being continuous on nearly all of its domain, in a form suited to our  $d$ -topology context.

**Theorem L (Lusin's Theorem;** cf. [Hal, end of Section 55]) *For  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable, there is a density-open set  $S$  which is almost all of  $\mathbb{R}_+$  and an increasing decomposition  $S := \bigcup_m S_m$  into density-open sets  $S_m$  such that each  $f|_{S_m}$  is continuous in the usual sense.*

*Proof* By a theorem of Lusin (see e.g. [Hal]), there is an increasing sequence of (non-null) compact sets  $K_n$  ( $n = 1, 2, \dots$ ) covering almost all of  $\mathbb{R}_+$  with the function  $f$  restricted to  $K_n$  continuous on  $K_n$ . Let  $S_n$  comprise the density points of  $K_n$ , so  $S_n$  is density-open, is almost all of  $K_n$  (by the Lebesgue Density Theorem) and  $f$  is continuous on  $S_n$ . Put  $S := \bigcup_m S_m$ ; then  $S$  is almost all of  $\mathbb{R}_+$ , is density-open and  $f|_{S_m}$  is continuous for each  $m$ .  $\square$

Two results, Theorem S below and Theorem 3.1 in the next section, depend on the following consequence of Steinhaus's theorem concerning the existence of interior points of  $A \cdot B^{-1}$  ([St], cf. [BGT, Th. 1.1.1]) for  $A, B$  measurable non-null. The first is in multiplicative form a sharper version of Sierpiński's result that any two non-null measurable sets realize a rational distance.

**Lemma S (Multiplicative Sierpiński Lemma;** [Sier]) *For  $a, b$  density points of their respective measurable sets  $A, B$  in  $\mathbb{R}_+$  and for  $n = 1, 2, \dots$ , there exist positive rationals  $q_n$  and points  $a_n, b_n$  converging to  $a, b$  through  $A, B$  respectively such that  $b_n = q_n a_n$ .*

*Proof* For  $n = 1, 2, \dots$  and the consecutive values  $\varepsilon = 1/n$  the sets  $B_\varepsilon(a) \cap A$  and  $B_\varepsilon(b) \cap B$  are measurable non-null, so by Steinhaus's theorem the set  $[B \cap B_\varepsilon(b)] \cdot [A \cap B_\varepsilon(a)]^{-1}$  contains interior points and so in particular a rational point  $q_n$ . Thus for some  $a_n \in B_\varepsilon(a) \cap A$



and  $b_n \in B_\varepsilon(b) \cap B$  we have  $q_n = b_n a_n^{-1}$ , and as  $|a - a_n| < 1/n$  and  $|b - b_n| < 1/n$ ,  $a_n \rightarrow a$ ,  $b_n \rightarrow b$ .  $\square$

**Remarks** 1. For the purposes of Theorem 3.2 below, we observe that  $q_n$  may be selected arbitrarily large, for fixed  $a$ , by taking  $b$  sufficiently large (since  $q_n \rightarrow ba^{-1}$ ).

2. The Lemma addresses  $d$ -open sets but also holds in the metric topology (the proof is similar but simpler), and so may be restated bitopologically (from the viewpoint of [BOst7]) as follows.

**Theorem S** (Sierpiński, [Sier]) *For  $\mathbb{R}_+$  with either the Euclidean or the density topology, if  $a, b$  are respectively in the open sets  $A, B$ , then for  $n = 1, 2, \dots$  there exist positive rationals  $q_n$  and points  $a_n, b_n$  converging metrically to  $a, b$  through  $A, B$  respectively such that*

$$b_n = q_n a_n.$$

### 3 A bitopological Kingman theorem

We begin by simplifying essential unboundedness modulo null/meagre sets.

**Theorem 3.0** *In  $\mathbb{R}_+$  with the Euclidean or density topology, for  $S$  Baire/measurable and essentially unbounded there exists an open/density-open unbounded  $G$  and meagre/null  $M$  with  $G \setminus M \subset S$ .*

*Proof* Choose integers  $m_n$  inductively with  $m_0 = 0$  and  $m_{n+1} > m_n$  the least integer such that  $(m_n, m_{n+1}) \cap S$  is non-meagre; for given  $m_n$  the integer  $m_{n+1}$  is well-defined, as otherwise for each  $m > m_n$  we would have  $(m_n, m) \cap S$  meagre, and so also

$$(m_n, \infty) \cap S = \bigcup_{m > m_n} (m_n, m) \cap S \text{ meagre,}$$

contradicting  $S$  essentially unbounded. Now, as  $(m_n, m_{n+1}) \cap S$  is Baire/measurable, we may choose  $G_n$  open/density-open and  $M_n, M'_n$  meagre subsets of  $(m_n, m_{n+1})$  such that

$$((m_n, m_{n+1}) \cap S) \cup M_n = G_n \cup M'_n.$$

Hence  $G_n$  is non-empty. Put  $G := \bigcup_n G_n$  and  $M := \bigcup_n M_n$ . Then  $M$  is meagre and  $G$  is open unbounded and, since  $M \cap (m_n, m_{n+1}) = M_n$

and  $G \cap (m_n, m_{n+1}) = G_n$ ,

$$G \setminus M = \bigcup_n G_n \setminus M = \bigcup_n G_n \setminus M_n \subset \bigcup_n (m_n, m_{n+1}) \cap S = S,$$

as asserted.  $\square$

**Definition (Weakly Archimedean property—an admissibility condition)** Let  $\mathbb{I}$  be  $\mathbb{N}$  or  $\mathbb{Q}$ , with the ordering induced from the reals. Our purpose will be to take limits through subsets  $J$  of  $\mathbb{I}$  which are *unbounded* on the right (more briefly: unbounded). According as  $\mathbb{I}$  is  $\mathbb{N}$  or  $\mathbb{Q}$ , we will write  $n \rightarrow \infty$  or  $q \rightarrow \infty$ . Denote by  $X$  the line with either the metric or density topology and say that a family  $\{h_i : i \in \mathbb{I}\}$  of self-homeomorphisms of the topological space  $X$  is *weakly Archimedean* if for each non-empty open set  $V$  in  $X$  and any  $j \in \mathbb{I}$  the open set

$$U_j(V) := \bigcup_{i \geq j} h_i(V)$$

meets every essentially unbounded set in  $X$ .

**Theorem 3.1** (implicit in [BGT, Th. 1.9.1(i)]) *In the multiplicative group of positive reals  $\mathbb{R}_+^*$  with Euclidean topology, the functions  $h_n(x) = d_n x$  for  $n = 1, 2, \dots$ , are homeomorphisms and  $\{h_n : n \in \mathbb{N}\}$  is weakly Archimedean, if  $d_n$  is divergent and the multiplicative form of (\*) holds. For any interval  $J = (a, b)$  with  $0 < a < b$  and any  $m$ ,*

$$U_m(J) := \bigcup_{n \geq m} d_n J$$

*contains an infinite half-line, and so meets every unbounded open set. Similarly this is the case in the additive group of reals  $\mathbb{R}$  with  $h_n(x) = d_n + x$  and  $U_m(J) = \bigcup_{n \geq m} (d_n + J)$ .*

*Proof* For given  $\varepsilon > 0$  and all large enough  $n$ ,  $1 - \varepsilon < d_n/d_{n+1} < 1 + \varepsilon$ . Write  $x := (a+b)/2 \in J$ . For  $\varepsilon$  small enough  $a < x(1-\varepsilon) < x(1+\varepsilon) < b$ , and then  $a < x d_n/d_{n+1} < b$ , so  $x d_n \in d_{n+1} J$ , and so  $d_n J$  meets  $d_{n+1} J$ . Thus for large enough  $n$  consecutive  $d_n J$  overlap; as  $d_n \rightarrow \infty$ , their union is thus a half-line.  $\square$

**Remark** Some such condition as (\*) is necessary, otherwise the set  $U_m(J)$  risks missing an unbounded sequence of open intervals. For an indirect example, see the remark in [BGT] after Th. 1.9.2 and G. E. H. Reuter's elegant counterexample to a corollary of Kingman's Theorem, a break-down caused by the absence of our condition. For a direct example, note that if  $d_n = r^n \log n$  with  $r > 1$  and  $J = (0, 1)$ , then  $d_n + J$  and

$d_{n+1} + J$  miss the interval  $(1 + r^n \log n, r^{n+1} \log(1 + n))$  and the omitted intervals have union an unbounded open set; to see that the omitted intervals are non-degenerate note that their lengths are unbounded:

$$r^{n+1} \log(1 + n) - r^n \log n - 1 \rightarrow \infty.$$

Theorem 3.1 does not extend to the real line under the density topology; the homeomorphisms  $h_n(x) = nx$  are no longer weakly Archimedean, as we demonstrate by an example in Theorem 4.6. We are thus led to an alternative approach:

**Theorem 3.2** *In the multiplicative group of reals  $\mathbb{R}_+^*$  with the density topology, the family of homeomorphisms  $\{h_q : q \in \mathbb{Q}_+\}$  defined by  $h_q(x) := qx$ , where  $\mathbb{Q}_+$  has its natural order, is weakly Archimedean. For any density-open set  $A$  and any  $j \in \mathbb{Q}_+$ ,*

$$U_j(A) := \bigcup_{q \geq j, q \in \mathbb{Q}_+} qA$$

*contains almost all of an infinite half-line, and so meets every unbounded density-open set.*

*Proof* Let  $B$  be Baire and essentially unbounded in the  $d$ -topology. Then  $B$  is measurable and essentially unbounded in the sense of measure. From Theorem 3.0, we may assume that  $B$  is density-open. Let  $A$  be non-empty density-open. Fix  $a \in A$  and  $j \in \mathbb{Q}_+$ . Since  $B$  is unbounded, we may choose  $b \in B$  such that  $b > ja$ . By Theorem S there is a  $q \in \mathbb{Q}_+$  with  $j < q < ba^{-1}$  such that  $qa' = b'$ , with  $a' \in A$  and  $b' \in B$ . Thus

$$U_j(A) \cap B \supseteq h_q(A) \cap B = qA \cap B \neq \emptyset,$$

as required.

If  $U_j(A)$  fails to contain almost all of any infinite half-line, then its complement  $B := \mathbb{R}_+ \setminus U_j(A)$  is essentially unbounded in the sense of measure and so, as above, must meet  $U_j(A)$ , a contradiction.  $\square$

- Remarks** 1. For  $A$  the set of irrationals in  $(0, 1)$  the set  $U_j(A)$  is again a set of irrationals which contains almost all, but not all, of an infinite half-line. Our result is thus best possible.
2. Note that  $(*)$  is relevant to the distinction between integer and rational skeletons; see the prime-divisor example on p. 53 of [BGT]. Theorem 3.2 holds with  $\mathbb{Q}_+$  replaced by any countable *dense* subset of  $\mathbb{R}_+^*$ , although later we use the fact that  $\mathbb{Q}_+$  is closed under

multiplication. There is an affinity here with the use of a dense ‘skeleton set’ in the Heiberg–Seneta Theorem, Th. 1.4.3 of [BGT], and its extension Th. 3.2.5 therein.

Kingman’s Theorem below, like KBD, is a generic assertion about embedding into target sets. We address first the source of this genericity: a property inheritable by supersets either holds generically or fails outright. This is now made precise.

**Definition** Recall that  $X$  denotes  $\mathbb{R}_+$  with Euclidean or density topology. Denote by  $\mathcal{Ba}(X)$ , or just  $\mathcal{Ba}$ , the Baire sets of the space  $X$ , and recall these form a  $\sigma$ -algebra. Say that a correspondence  $F : \mathcal{Ba} \rightarrow \mathcal{Ba}$  is *monotonic* if  $F(S) \subseteq F(T)$  for  $S \subseteq T$ .

The nub is the following simple result, which we call the Generic Dichotomy Principle.

**Theorem 3.3 (Generic Dichotomy Principle)** *For  $F : \mathcal{Ba} \rightarrow \mathcal{Ba}$  monotonic: either*

- (i) *there is a non-meagre  $S \in \mathcal{Ba}$  with  $S \cap F(S) = \emptyset$ , or*
- (ii) *for every non-meagre  $T \in \mathcal{Ba}$ ,  $T \cap F(T)$  is quasi all of  $T$ .*

*Equivalently: the existence condition that  $S \cap F(S) \neq \emptyset$  should hold for all non-meagre  $S \in \mathcal{Ba}$  implies the genericity condition that, for each non-meagre  $T \in \mathcal{Ba}$ ,  $T \cap F(T)$  is quasi all of  $T$ .*

*Proof* Suppose that (i) fails. Then  $S \cap F(S) \neq \emptyset$  for every non-meagre  $S \in \mathcal{Ba}$ . We show that (ii) holds. Suppose otherwise; thus for some  $T$  non-meagre in  $\mathcal{Ba}$ , the set  $T \cap F(T)$  is not almost all of  $T$ . Then the set  $U := T \setminus F(T) \subseteq T$  is non-meagre (it is in  $\mathcal{Ba}$  as  $T$  and  $F(T)$  are) and so

$$\begin{aligned} \emptyset \neq U \cap F(U) & \quad (S \cap F(S) \neq \emptyset \text{ for every non-meagre } S) \\ & \subseteq U \cap F(T) \quad (U \subseteq T \text{ and } F \text{ monotonic}). \end{aligned}$$

But as  $U := T \setminus F(T)$ ,  $U \cap F(T) = \emptyset$ , a contradiction.

The final assertion simply rephrases the dichotomy as an implication.  $\square$

The following corollary transfers the onus of verifying the existence condition of Theorem 3.3 to topological completeness.

**Theorem 3.4 (Generic Completeness Principle)** *For  $F : \mathcal{Ba} \rightarrow \mathcal{Ba}$  monotonic, if  $W \cap F(W) \neq \emptyset$  for all non-meagre  $W \in \mathcal{G}_\delta$  then, for each non-meagre  $T \in \mathcal{Ba}$ ,  $T \cap F(T)$  is quasi all of  $T$ .*

That is, either

- (i) there is a non-meagre  $S \in \mathcal{G}_\delta$  with  $S \cap F(S) = \emptyset$ , or
- (ii) for every non-meagre  $T \in \mathcal{B}a$ ,  $T \cap F(T)$  is quasi all of  $T$ .

*Proof* From Theorem B, for  $S$  non-meagre in  $\mathcal{B}a$  there is a non-meagre  $W \subseteq S$  with  $W \in \mathcal{G}_\delta$ . So  $W \cap F(W) \neq \emptyset$  and thus  $\emptyset \neq W \cap F(W) \subseteq S \cap F(S)$ , by monotonicity. By Theorem 3.3 for every non-meagre  $T \in \mathcal{B}a$ ,  $T \cap F(T)$  is quasi all of  $T$ .  $\square$

**Remarks** In regard to the role of  $\mathcal{G}_\delta$  sets, we note *Solecki's analytic dichotomy theorem* (reformulating and generalizing a specific instance discovered by Petruska, [Pet]) as follows. For  $\mathcal{I}$  a family of closed sets (in any Polish space), let  $\mathcal{I}_{\text{ext}}$  denote the sets covered by a countable union of sets in  $\mathcal{I}$ . Then, for  $A$  an analytic set, either  $A \in \mathcal{I}_{\text{ext}}$ , or  $A$  contains a  $\mathcal{G}_\delta$  set not in  $\mathcal{I}_{\text{ext}}$ . See [Sol1], where a number of classical theorems, asserting that a ‘large’ analytic set contains a ‘large’ compact subset, are deduced, and also [Sol2] for further applications of dichotomy. A superficially similar, but more distant result, is *Kuratowski's Dichotomy*—([Kur-B], [Kur-1], [McSh, Cor. 1]): suppose a set  $H$  of auto-homeomorphisms acts transitively on a space  $X$ , and  $Z \subseteq X$  is Baire and has the property that for each  $h \in H$

$$Z = h(Z) \text{ or } Z \cap h(Z) = \emptyset,$$

i.e. under each  $h \in H$ , either  $Z$  is invariant or  $Z$  and its image are disjoint. Then, either  $H$  is meagre or it is clopen.

**Examples** Here are four examples of monotonic correspondences. The first two relate to standard results. The following two are canonical for the current paper as they relate to KBD and to Kingman's Theorem in its original form. Each correspondence  $F$  below gives rise to a correspondence  $\Phi(A) := F(A) \cap A$  which is a lower or upper density and arises in the theory of *lifting* [IT1, IT2] and category measures [Oxt, Th. 22.4], and so gives rise to a fine topology on the real line. See also [LMZ, Section 6F] on lifting topologies.

1. Here we apply Theorem 3.3 to the real line with the density topology, in which the meagre sets are the null sets. Let  $\mathcal{B}$  denote a countable basis of Euclidean open neighbourhoods. For any set  $T$  and  $0 < \alpha < 1$  put

$$\mathcal{B}_\alpha(T) := \{I \in \mathcal{B} : |I \cap T| > \alpha|I|\},$$

which is countable, and

$$F(T) := \bigcap_{\alpha \in \mathbb{Q} \cap (0,1)} \bigcup \{I : I \in \mathcal{B}_\alpha(T)\}.$$

Thus  $F$  is increasing in  $T$ ,  $F(T)$  is measurable (even if  $T$  is not) and  $x \in F(T)$  iff  $x$  is a density point of  $T$ . If  $T$  is measurable, the set of points  $x$  in  $T$  for which  $x \in I \in \mathcal{B}$  implies that  $|I \cap T| < \alpha|I|$  is null (see [Oxt, Th. 3.20]). Hence any non-null measurable set contains a density point. It follows that almost all points of a measurable set  $T$  are density points. This is the Lebesgue Density Theorem ([Oxt, Th. 3.20], or [Kucz, Section 3.5]).

2. In [PWW, Th. 2] a category analogue of the Lebesgue Density Theorem is established. This follows more simply from our Theorem 3.3.
3. For KBD, let  $z_n \rightarrow 0$  and put  $F(T) := \bigcap_{n \in \omega} \bigcup_{m > n} (T - z_m)$ . Thus  $F(T) \in \mathcal{B}a$  for  $T \in \mathcal{B}a$  and  $F$  is monotonic. Here  $t \in F(T)$  iff there is an infinite  $\mathbb{M}_t$  such that  $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$ . The Generic Dichotomy Principle asserts that once we have proved (for which see Theorem 5.3 below) that an arbitrary non-meagre set  $T$  contains a ‘translator’, i.e. an element  $t$  which shift-embeds a subsequence  $z_m$  into  $T$ , then quasi all elements of  $T$  are translators.
4. For  $z_n = n$  and  $\{S_k\}$  a family of unbounded open sets (in the Euclidean sense), put  $F(T) := T \cap \bigcap_{k \in \omega} \bigcap_{n \in \omega} \bigcup_{m > n} (S_k - z_m)$ . Thus  $F(T) \in \mathcal{B}a$  for  $T \in \mathcal{B}a$  and  $F$  is monotonic. Here  $t \in F(T)$  iff  $t \in T$  and for each  $k \in \omega$ , there is an infinite  $\mathbb{M}_t^k$  such that  $\{t + z_m : m \in \mathbb{M}_t^k\} \subseteq S_k$ . In Kingman’s version of his theorem, as stated, we know only that  $F(V)$  is non-empty for any non-empty open set  $V$ ; but in Theorem 3.5 below we adjust his argument to show that  $F(T)$  is non-empty for arbitrary non-meagre sets  $T \in \mathcal{B}a$ , hence that quasi all members of  $T$  are in  $F(T)$ , and in particular that this is so for  $T = \mathbb{R}_+$ .

**Theorem 3.5 (Bitopological Kingman Theorem—**[King1, Th. 1], [King2], where  $\mathbb{I} = \mathbb{N}$ ) *If  $X$  is a Baire space,*

- (i)  *$\{h_i : i \in \mathbb{I}\}$  is a countable, linearly ordered, weakly Archimedean family of self-homeomorphisms of  $X$ , and*
- (ii)  *$\{S_k : k = 1, 2, \dots\}$  are essentially unbounded Baire sets,*

*then for quasi all  $\eta \in X$  and all  $k \in \mathbb{N}$  there exists an unbounded subset  $\mathbb{J}_\eta^k$  of  $\mathbb{I}$  with*

$$\{h_j(\eta) : j \in \mathbb{J}_\eta^k\} \subset S_k.$$

Equivalently, if (i) and

(i)'  $\{A_k : k = 1, 2, \dots\}$  are Baire and all accumulate essentially at 0,

then for quasi all  $\eta$  and every  $k = 1, 2, \dots$  there exists  $\mathbb{J}_\eta^k$  unbounded with

$$\{h_j(\eta)^{-1} : j \in \mathbb{J}_\eta^k\} \subset A_k.$$

*Proof* We will apply Theorem 3.3 (Generic Dichotomy), so consider an arbitrary non-meagre Baire set  $T$ . We may assume without loss of generality that  $T = V \setminus M$  with  $V$  non-empty open and  $M$  meagre. For each  $k = 1, 2, \dots$  choose  $G_k$  open and  $N_k$  and  $N'_k$  meagre such that  $S_k \cup N'_k = G_k \cup N_k$ . Put  $N := M \cup \bigcup_{n,k} h_n^{-1}(N'_k)$ ; then  $N$  is meagre (as  $h_n$ , and so  $h_n^{-1}$ , is a homeomorphism).

As  $S_k$  is essentially unbounded,  $G_k$  is unbounded (otherwise, for some  $m$ ,  $G_k \subset (-m, m)$ , and so  $S_k \cap (m, \infty) \subset N_k$  is meagre). Define the open sets  $G_{jk} := \bigcup_{i \geq j} h_i^{-1}(G_k)$ . We first show that each  $G_{jk}$  is dense. Suppose, for some  $j, k$ , there is a non-empty open set  $V$  such that  $V \cap G_{jk} = \emptyset$ . Then for all  $i \geq j$ ,

$$V \cap h_i^{-1}(G_k) = \emptyset; \quad G_k \cap h_i(V) = \emptyset.$$

So  $G_k \cap \bigcup_{i \geq j} h_i(V) = \emptyset$ , i.e., for  $U^j$  the open set  $U^j := \bigcup_{i \geq j} h_i(V)$  we have  $G_k \cap U^j = \emptyset$ . But as  $G_k$  is unbounded, this contradicts  $\{h_i\}$  being a weakly Archimedean family.

Thus the open set  $G_{jk}$  is dense (meets every non-empty open set); so, as  $\mathbb{I}$  is countable, the  $\mathcal{G}_\delta$  set

$$H := \bigcap_{k=1}^{\infty} \bigcap_{j \in \mathbb{I}} G_{jk}$$

is dense (as  $X$  is a Baire space). So as  $V$  is a non-empty open subset we may choose  $\eta \in (H \cap V) \setminus N$ . (Otherwise  $N \cup (X \setminus H)$  and hence  $V$  is of first category.) Thus  $\eta \in T$  and for all  $k = 1, 2, \dots$

$$\eta \in V \cap \bigcap_{j \in \mathbb{I}} \bigcup_{i \geq j} h_i^{-1}(G_k) \text{ and } \eta \notin N. \quad (\text{eta})$$

For all  $m$ , as  $h_m(\eta) \notin h_m(N)$  we have for all  $m, k$  that  $h_m(\eta) \notin N'_k$ . Using (eta), for each  $k$  select an unbounded  $\mathbb{J}_\eta^k$  such that for  $j \in \mathbb{J}_\eta^k$ ,  $\eta \in h_j^{-1}(G_k)$ ; for such  $j$  we have  $\eta \in h_j^{-1}(S_k)$ . That is, for some  $\eta \in T$  we have

$$\{h_j(\eta) : j \in \mathbb{J}_\eta^k\} \subset S_k.$$

Now

$$F(T) := T \cap \bigcap_{k=1}^{\infty} \bigcap_{j \in \mathbb{I}} \bigcup_{i \geq j} h_i^{-1}(G_k)$$

takes Baire sets to Baire sets and is monotonic. Moreover,  $\eta \in F(T)$  iff  $\eta \in T$  and for each  $k$  there is an unbounded  $\mathbb{J}_\eta^k$  with  $\{h_j(\eta) : j \in \mathbb{J}_\eta^k\} \subset S_k$ . We have just shown that  $T \cap F(T) \neq \emptyset$  for  $T$  arbitrary non-meagre, so the Generic Dichotomy Principle implies that  $X \cap F(X)$  is quasi all of  $X$ , i.e. for quasi all  $\eta$  in  $X$  and each  $k$  there is an unbounded  $\mathbb{J}_\eta^k$  with  $\{h_j(\eta) : j \in \mathbb{J}_\eta^k\} \subset S_k$ .  $\square$

Working in either the density or the Euclidean topology, we obtain the following conclusions.

**Theorem 3.6C (Kingman Theorem for Category)** *If  $\{S_k : k = 1, 2, \dots\}$  are Baire and essentially unbounded in the category sense, then for quasi all  $\eta$  and each  $k \in \mathbb{N}$  there exists an unbounded subset  $\mathbb{J}_\eta^k$  of  $\mathbb{N}$  with*

$$\{n\eta : n \in \mathbb{J}_\eta^k\} \subset S_k.$$

*In particular this is so if the sets  $S_k$  are open.*

**Theorem 3.6M (Kingman Theorem for Measure)** *If  $\{S_k : k = 1, 2, \dots\}$  are measurable and essentially unbounded in the measure sense, then for almost all  $\eta$  and each  $k \in \mathbb{N}$  there exists an unbounded subset  $\mathbb{J}_\eta^k$  of  $\mathbb{Q}_+$  with*

$$\{q\eta : q \in \mathbb{J}_\eta^k\} \subset S_k.$$

In the corollary below  $\mathbb{J}_t^k$  refers to unbounded subsets of  $\mathbb{N}$  or  $\mathbb{Q}_+$  according to the category/measurable context. It specializes down to a KBD result for a single set  $T$  when  $T_k \equiv T$ , but it falls short of KBD in view of the extra admissibility assumption and the factor  $\sigma$  (the latter an artefact of the multiplicative setting).

**Corollary** *For  $\{T_k : k \in \omega\}$  Baire/measurable and  $z_n \rightarrow 0$  admissible, for generically all  $t \in \mathbb{R}$  there exist  $\sigma_t$  and unbounded  $\mathbb{J}_t^k$  such that for  $k = 1, 2, \dots$*

$$t \in T_k \implies \{t + \sigma_t z_m : m \in \mathbb{J}_t^k\} \subset T_k.$$

*Proof* For  $T$  Baire/measurable, let  $N = N(T)$  be the set of points  $t \in T$  that are not points of essential accumulation of  $T$ ; then  $t \in N$  if for some  $n = n(t)$  the set  $T \cap B_{1/n}(t)$  is meagre/null. As  $\mathbb{R}$  with the Euclidean topology is (hereditarily) second-countable, it is hereditarily



Lindelöf (see [Eng, Th. 3.8.1] or [Dug, Th. 8.6.3]), so for some countable  $S \subset N$

$$N \subset \bigcup_{t \in S} T \cap B_{1/n(t)}(t),$$

and so  $N$  is meagre/null. Thus the set  $N_k$  of points  $t \in T_k$  such that  $T_k - t$  does not accumulate essentially at 0 is meagre/null, as is  $N = \bigcup_k N_k$ . For  $t \notin N$ , put  $\Omega_t := \{k \in \omega : T_k - t \text{ accumulates essentially at } 0\}$ . Applying Kingman's Theorem to the sets  $\{T_k - t : k \in \Omega_t\}$  and the sequence  $z_n \rightarrow 0$ , there exist  $\sigma_t$  and unbounded  $\mathbb{J}_t^k$  such that for  $k \in \Omega_t$

$$\{\sigma_t z_m : m \in \mathbb{J}_t^k\} \subset T_k - t, \text{ i.e. } \{t + \sigma_t z_m : m \in \mathbb{J}_t^k\} \subset T_k.$$

Thus for  $t \notin N$ , so for generically all  $t$ , there exist  $\sigma_t$  and unbounded  $\mathbb{J}_t^k$  such that for  $k = 1, 2, \dots$

$$t \in T_k \implies \{t + \sigma_t z_m : m \in \mathbb{J}_t^k\} \subset T_k. \quad \square$$

## 4 Applications—rational skeletons

In [King1] Kingman's applications were concerned mostly with limiting behaviour of continuous functions, studied by means of  $h$ -skeletons defined by

$$L_{\mathbb{N}}(h) := \lim_{n \rightarrow \infty} f(nh),$$

assumed to exist for all  $h$  in some interval  $I$ . This works for Baire functions; but in our further generalization to measurable functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we are led to study limits  $L_{\mathbb{Q}}(h) := \lim_{q \rightarrow \infty} f(qh)$ , taken through the rationals. Using the decomposition

$$q := n(q) + r(q), \quad n(q) \in \omega, \quad r(q) \in [0, 1),$$

the limit  $L_{\mathbb{Q}}(h)$  may be reduced to, and so also computed as,  $L_{\mathbb{N}}(h)$  (provided we admit perturbations on  $h$ , making the assumption of convergence here more demanding)—see Theorem 4.5 below.

**Theorem 4.1 (Conversion of sequential to continuous limits at infinity—cf. [King1, Cor. 2 to Th. 1])** *For  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable and  $V$  a non-empty, density-open set (in particular, an open interval), if*

$$\lim_{q \rightarrow \infty} f(qx) = 0, \text{ for each } x \in V,$$

then

$$\lim_{t \rightarrow \infty} f(t) = 0.$$

The category version holds, *mutatis mutandis*, with  $f$  Baire and  $V$  open.

*Proof* Suppose not; choose  $c > 0$  with  $\limsup_{t \rightarrow \infty} |f(t)| > c > 0$ . By Theorem L there is a density-open set  $S$  which is almost all of  $\mathbb{R}_+$  and an increasing decomposition  $S := \bigcup_m S_m$  such that each  $f|_{S_m}$  is continuous. Put  $B = \{s \in S : |f(s)| > c\}$ . For any  $M > 0$ , there is an  $s^* \in S_m$  for some  $m$  with  $s^* > M$  such that  $|f(s^*)| > c$ . Then by continuity of  $f|_{S_m}$ , for some  $\delta > 0$  we have  $|f(s)| > c$  for  $s \in B_\delta(s^*) \cap S_m$ . Thus  $B$  is essentially unbounded. By Theorem 3.6M there exists  $v \in V$  such that  $qv \in B$ , for unboundedly many  $q \in \mathbb{Q}_+$ ; but, for such a  $v$ , we have  $\lim_{q \rightarrow \infty} f(qv) \neq 0$ , a contradiction.  $\square$

We will need the following result, which is of independent interest. The Baire case is implicit in [King1, Th. 2]. A related Baire category result is in [HJ, Th. 2.3.7] (with  $G = \mathbb{R}_+$  and  $T = \mathbb{Q}$  there).

**Theorem 4.2 (Constancy of rationally invariant functions)** *If for  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable*

$$f(qx) = f(x), \text{ for } q \in \mathbb{Q}_+ \text{ and almost all } x \in \mathbb{R}_+,$$

*then  $f(x)$  takes a constant value almost everywhere. The category version holds, *mutatis mutandis*, with  $f$  Baire.*

*Proof (for the measure case)* Again by Theorem L, there is a density-open set  $S$  which is almost all of  $\mathbb{R}_+$  and an increasing decomposition  $S := \bigcup_m S_m$  such that each  $f|_{S_m}$  is continuous. We may assume without loss of generality that  $f(qx) = f(x)$ , for  $q \in \mathbb{Q}_+$  and all  $x \in S$ . We claim that on  $S$  the function  $f$  is constant. Indeed, by Theorem S if  $a, b$  are in  $S_m$ , then, since they are density points of  $S_m$ , there are  $a_n, b_n$  in  $S_m$  and  $q_n$  in  $\mathbb{Q}_+$  such that  $b_n = q_n a_n$  with  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , as  $n \rightarrow \infty$  (in the metric sense). Hence, since  $f(q_n a_n) = f(a_n)$ , relative continuity gives

$$f(b) = \lim_{n \rightarrow \infty} f(b_n) = \lim_{n \rightarrow \infty} f(q_n a_n) = \lim_{n \rightarrow \infty} f(a_n) = f(a).$$

The Baire case is similar, but simpler.  $\square$

Of course, if  $f(x)$  is the indicator function  $1_{\mathbb{Q}}(x)$ , which is measurable/Baire, then  $f(x)$  is constant almost everywhere, but not constant, so the result in either setting is best possible.

**Theorem 4.3 (Uniqueness of limits—cf. [King1, Th. 2])** For  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable, suppose that for each  $x > 0$  the limit

$$L(x) := \lim_{q \rightarrow \infty} f(qx)$$

exists and is finite on a density-open set  $V$  (in particular an interval). Then  $L(x)$  takes a constant value a.e. in  $\mathbb{R}_+$ ,  $L$  say, and

$$\lim_{t \rightarrow \infty} f(t) = L.$$

The category version holds, *mutatis mutandis*, with  $f$  Baire and  $V$  open.

*Proof* Since  $\mathbb{Q}_+$  is countable, the function  $L(x)$  is measurable/Baire. Note that for  $q \in \mathbb{Q}_+$  one has  $L(qx) = L(x)$ , so if  $L$  is defined for  $x \in V$ , then  $L$  is defined for  $x \in qV$  for each  $q \in \mathbb{Q}_+$ , so by Theorem 3.2 for almost all  $x$  in some half-infinite interval and thus for almost all  $x \in \mathbb{R}_+$ . The result now follows from Theorem 4.2. As for the final conclusion, replacing  $f(x)$  by  $f(x) - L$ , we may suppose that  $L = 0$ , and so may apply Theorem 4.1.  $\square$

We now extend an argument in [King1]. Recall that  $f$  is *essentially bounded* on  $S$  if  $\text{ess sup}_S f < \infty$ .

**Definition** Call  $f$  *essentially bounded at infinity* if for some  $M$   $\text{ess sup}_{(n,\infty)} f \leq M$ , for all large enough  $n$ , i.e.  $\limsup_{n \rightarrow \infty} [\text{ess sup}_{(n,\infty)} f] < \infty$ .

**Theorem 4.4 (Essential boundedness theorem—cf. [King1, Cor. 3 to Th. 1])** For  $V$  non-empty density-open and  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable, suppose that for each  $x \in V$

$$\sup\{f(qx) : q \in \mathbb{Q}_+\} < \infty.$$

Then  $f(t)$  is essentially bounded at infinity. The category version holds, *mutatis mutandis*, with  $f$  Baire and  $V$  open.

*Proof* Suppose not. Then for each  $n = 1, 2, \dots$  there exists  $m \in \mathbb{N}$ , arbitrarily large, such that  $\text{ess sup}_{(m,\infty)} f > n$ . We proceed inductively. Suppose that  $m(n)$  has been defined so that  $\text{ess sup}_{(m(n),\infty)} f > n$ . Choose  $m(n+1) > m(n)$  so that

$$G_n := \{t : m(n) < t < m(n+1) \text{ and } |f(t)| > n\}$$

is non-null (otherwise, off a null set, we would have  $|f(t)| \leq n$  for all  $t > m(n)$ , making  $n$  an essential bound of  $f$  on  $(m, \infty)$ , for each  $m >$

$m(n)$ , contradicting the assumed essential unboundedness). Each  $G_n$  is measurable non-null, so defining

$$G := \bigcup_n G_n$$

yields  $G$  essentially unbounded. So there exists  $v \in V$  such that  $qv \in G$ , for an unbounded set of  $q \in \mathbb{Q}_+$ . Since each set  $G_n$  is bounded, the set  $\{qv : q \in \mathbb{Q}_+\}$  meets infinitely many of the disjoint sets  $G_n$ , and so

$$\sup\{|f(qv)| : q \in \mathbb{Q}_+\} = \infty,$$

contradicting our assumption.  $\square$

We close with the promised comparison of  $L_{\mathbb{Q}}(h)$  with  $L_{\mathbb{N}}(h)$  (of course, if  $L_{\mathbb{Q}}(h)$  exists, then so does  $L_{\mathbb{N}}(h)$  and they are equal). We use the decomposition  $q = n(q) + r(q)$ , with  $n(q) \in \mathbb{N}$  and  $r(q) \in [0, 1) \cap \mathbb{Q}$ .

**Theorem 4.5 (Perturbed skeletons)** *For  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  measurable, the limit  $L_{\mathbb{Q}}(h)$  exists for all  $h$  in the non-empty interval  $I = (0, b)$  with  $b > 0$  iff for all  $h$  in  $I$  the limit*

$$\lim_n f(n(h + z_n))$$

*exists, for every null sequence  $z_n$  with*

$$r_n := n(z_n/h) \in [0, 1) \cap \mathbb{Q}.$$

*Furthermore, if either limit exists on  $I$  then it exists a.e. on  $\mathbb{R}_+$ , and then both limits are equal a.e. to  $L_{\mathbb{N}}(h)$ .*

*If  $I = (a, b)$  with  $0 < a < b$ , the assertion holds far enough to the right.*

*Proof* First we prove the asserted equivalence.

Suppose the limit  $L_{\mathbb{Q}}(h)$  exists for all  $h$  in the interval  $I$ . Then given  $z_n$  as above, take  $q_n := n + r_n$ ,  $r_n = n(z_n/h)$ ; then  $n(q_n) = n$ ,  $r(q_n) = r_n$ , and

$$q_n h = n(h + z_n).$$

So the following limit exists:

$$\lim_n f(n(h + z_n)) = \lim_n f(q_n h) = L_{\mathbb{Q}}(h).$$

For the converse, take  $z_n$  as above (so  $z_n = r_n h/n$  with  $r_n \in [0, 1) \cap \mathbb{Q}$ ); our assumption is that  $L(h, \{r_n\})$  exists for all  $h \in I$ , where

$$L(h, \{r_n\}) := \lim_n f(n(h + z_n)).$$

Write  $\partial L(\{r_n\})$  for the ‘domain of  $L$ ’—the set of  $h$  for which this limit exists. Thus  $I \subseteq \partial L(\{r_n\})$ . Let  $q_n \rightarrow \infty$  be arbitrary in  $\mathbb{Q}_+$ . So  $q_n = n(q_n) + r(q_n)$  and, writing  $r_n := r(q_n)$  and  $z_n := r_n h / n(q_n)$ , we find

$$q_n h = n(q_n)[h + r_n h / n(q_n)] = n(q_n)[h + z_n],$$

and

$$r_n = n(z_n / h) \in [0, 1) \cap \mathbb{Q}.$$

By assumption,  $\lim_n f(q_n h) = L(h, \{r_n\})$  exists for  $h \in I$ . Restricting from  $\{n\}$  to  $\{pn\}$ , the limit  $\lim_n f(np(h + z_n))$  exists for each  $p \in \mathbb{N}$ , and

$$L(h, \{r_n\}) = \lim_n f(np(h + z_n)) = \lim_n f(n(h' + z'_n)),$$

where  $z'_n = pz_n$  and  $h' = ph$ . So

$$L(h, \{r_n\}) = L(ph, \{r_n\}),$$

as

$$n(z'_n / h') = n(z_n / h) = r_n \in [0, 1) \cap \mathbb{Q}.$$

That is,  $L(ph, \{r_n\})$  exists for  $p$  a positive integer, whenever  $L(h, \{r_n\})$  exists, and equals  $L(h, \{r_n\})$ . As  $h/p \in I = (0, b)$  for  $h \in I$  and  $L(h, \{r_n\}) = L(p(h/p), \{r_n\}) = L(h/p, \{r_n\})$ ,  $L(rh, \{r_n\})$  exists whenever  $r$  is a positive rational. So the domain  $\partial L$  of  $L$  includes all intervals of the form  $rI$  for positive rational  $r$ , and so includes the whole of  $\mathbb{R}_+$ . Moreover,  $L(\cdot, \{r_n\})$  is rationally invariant. But, since  $f$  is measurable,  $L$  is a measurable function on  $I \times \mathbb{Q}_+^\omega$ . Now  $\mathbb{Q}_+$  can be identified with  $\mathbb{N} \times \mathbb{N}$ , and hence  $\mathbb{Q}_+^\omega$  can be identified with  $\mathbb{N}^\omega$ . This in turn may be identified with the irrationals  $\mathcal{I}$  (see e.g. [JR, p. 9]). So  $L$  is measurable on  $\mathbb{R}_+ \times \mathcal{I}$  and so, by Theorem 4.2,  $L(\cdot, \{r_n\})$  is almost everywhere constant.

This proves the equivalence asserted. For the final assertion, the argument in the last paragraph shows that, given the assumptions,  $L_{\mathbb{Q}}(h)$  exists for a.e. positive  $h$ ; from here the a.e. equality is immediate. This completes the case  $I = (0, b)$ . For  $I = (a, b)$  with  $a > 0$ ,  $\bigcup_{p \in \mathbb{N}} pI$  contains some half-line  $[c, \infty)$  by Theorem 3.1.  $\square$

The following example, due to R. O. Davies, clarifies why use of the natural numbers and hence also of discrete skeletons  $L_{\mathbb{N}}(h)$  is inadequate in the measure setting. (We thank Roy Davies for this contribution.)

**Theorem 4.6** *The open set*

$$G := \bigcup_{m=1}^{\infty} (m - 2^{-(m+2)}, m)$$

*is disjoint for each  $n = 1, 2, \dots$  from the dilation  $nF$  of the non-null closed set  $F$  defined by*

$$F := \left[ \frac{1}{2}, 1 \right] \setminus \left( \bigcup_{m=1}^{\infty} \bigcup_{n=m}^{2m-1} \left( \frac{m}{n} - \frac{1}{n2^{m+2}}, \frac{m}{n} \right) \right).$$

*Proof* Suppose not. Put  $z_m := 2^{-(m+2)}$  and

$$E := \bigcup_{m=1}^{\infty} \bigcup_{n=m}^{2m-1} \left( \frac{m}{n} - \frac{1}{n} z_m, \frac{m}{n} \right).$$

Then for some  $n$ , there are  $f \in F$  and  $g \in G$  such that  $nf = g$ . So for some  $m = 1, 2, \dots$

$$m - z_m < nf = g < m, \text{ i.e. } \frac{m}{n} - \frac{z_m}{n} < f < \frac{m}{n}.$$

But as  $1/2 \leq f \leq 1$ , we have  $n/2 \leq m$  and

$$\frac{m}{n} - \frac{1}{n} z_m < 1, \text{ i.e. } m - z_m < n.$$

Thus  $m \leq n \leq 2m$ , yielding the contradiction that  $f \notin F$ . Put

$$a_m := \sum_{n=m}^{2m-1} \frac{1}{n}, \text{ so that } \frac{1}{2} \leq a_m \leq 1.$$

Then

$$|E| = \sum_{m=1}^{\infty} a_m z_m \leq \sum_{m=1}^{\infty} 2^{-(m+2)} = \frac{1}{4},$$

and  $|E| \geq 1/8$ . Hence the complementary set  $F$  has measure at least  $1/4$ .  $\square$

## 5 KBD in van der Waerden style

Fix  $p$ . Let  $z_n$  be a null sequence. We prove a generalization of KBD inspired by the van der Waerden theorem on arithmetic progressions (see Section 6). For this we need the notation

$$t + \bar{z}_{pm} = t + z_{pm+1}, t + z_{pm+1}, \dots, t + z_{pm+p}$$

as an abbreviation for a block of consecutive terms of the null sequence all shifted by  $t$ . Our unified proof, based on the  $\mathcal{G}_\delta$ -inner regularity common to measure and category noted in Section 2, is inspired by a technique in [BHW].

**Theorem 5.1 (Kestelman–Borwein–Ditor Theorem—consecutive form; [BOst5])** *Let  $\{z_n\} \rightarrow 0$  be a null sequence of reals. If  $T$  is Baire/Lebesgue-measurable and  $p \in \mathbb{N}$ , then for generically all  $t \in T$  there is an infinite set  $\mathbb{M}_t$  such that*

$$\{t + \bar{z}_{pm} : m \in \mathbb{M}_t\} := \{t + z_{pm+1}, t + z_{pm+1}, \dots, t + z_{pm+p} : m \in \mathbb{M}_t\} \subseteq T.$$

This will follow from the two results below, both important in their own right. The first and its corollary address displacements of open sets in the density and the Euclidean topologies; it is mentioned in passing in a note added in proof (p. 32) in Kemperman [Kem, Th. 2.1, p. 30], for which we give an alternative proof. The second parallels an elegant result for the measure case treated in [BHW].

**Theorem K (Displacements Lemma—Kemperman’s Theorem; [Kem, Th. 2.1] with  $B_i = E$ ,  $a_i = t$ )** *If  $E$  is non-null Borel, then  $f(x) := |E \cap (E + x)|$  is continuous at  $x = 0$ , and so for some  $\varepsilon = \varepsilon(E) > 0$*

$$E \cap (E + x) \text{ is non-null, for } |x| < \varepsilon.$$

*More generally,  $f(x_1, \dots, x_p) := |(E + x_1) \cap \dots \cap (E + x_p)|$  is continuous at  $x = (0, \dots, 0)$ , and so for some  $\varepsilon = \varepsilon_p(E) > 0$*

$$(E + x_1) \cap \dots \cap (E + x_p) \text{ is non-null, for } |x_i| < \varepsilon \quad (i = 1, \dots, p).$$

*Proof 1* (After [BHW]; cf. e.g. [BOst6, Th. 6.2 and 7.5].) Let  $t$  be a density point of  $E$ . Choose  $\varepsilon > 0$  such that

$$|E \cap B_\varepsilon(t)| > \frac{3}{4}|B_\varepsilon(0)|.$$

Now  $|B_\varepsilon(t) \setminus B_\varepsilon(t+x)| \leq (1/4)|B_\varepsilon(t+x)|$  for  $x \in B_{\varepsilon/2}(0)$ , so

$$|E \cap B_\varepsilon(t+x)| > \frac{1}{2}|B_\varepsilon(0)|.$$

By invariance of Lebesgue measure we have

$$|(E+x) \cap B_\varepsilon(t+x)| > \frac{3}{4}|B_\varepsilon(0)|.$$

But, again by invariance, as  $B_\varepsilon(t)+x = B_\varepsilon(0)+t+x$  this set has measure

$|B_\varepsilon(0)|$ . Using  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$  with  $A_1 := E \cap B_\varepsilon(t+x)$  and  $A_2 := (E+x) \cap B_\varepsilon(t+x)$  now yields

$$|E \cap (E+x)| \geq |E \cap (E+x) \cap (B_\varepsilon(t) + x)| > \frac{5}{4}|B_\varepsilon(0)| - |B_\varepsilon(0)| > 0.$$

Hence, for  $x \in B_{\varepsilon/2}(0)$ , we have  $|E \cap (E+x)| > 0$ .

For the  $p$ -fold form we need some notation. Let  $t$  again denote a density point of  $E$  and  $x = (x_1, \dots, x_n)$  a vector of variables. Set  $A_j := B(t) \cap E \cap (E+x_j)$  for  $1 \leq j \leq n$ . For each multi-index  $\mathbf{i} = (i(1), \dots, i(d))$  with  $0 < d < n$ , put

$$\begin{aligned} f_{\mathbf{i}}(x) &:= |A_{i(1)} \cap \dots \cap A_{i(d)}|; \\ f_n(x) &:= |A_1 \cap \dots \cap A_n|, \quad f_0 = |B(t) \cap E|. \end{aligned}$$

We have already shown that the functions  $f_j(x) = |B(t) \cap E \cap (E+x_j)|$  are continuous at 0. Now argue inductively: suppose that, for  $\mathbf{i}$  of length less than  $n$ , the functions  $f_{\mathbf{i}}$  are continuous at  $(0, \dots, 0)$ . Then for given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for  $\|x\| < \delta$  and each such index  $\mathbf{i}$  we have

$$-\varepsilon < f_{\mathbf{i}}(x) - f_0 < \varepsilon,$$

where  $f_0 = |B(t) \cap E|$ . Noting that

$$\bigcup_{i=1}^n A_i \subset B(t) \cap E,$$

and using upper or lower approximations, according to the signs in the inclusion-exclusion identity

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_i |A_i| - \sum_{i < j} |A_i \cap A_j| + \dots + (-1)^{n-1} \left| \bigcap_i A_i \right|,$$

one may compute linear functions  $L(\varepsilon)$ ,  $R(\varepsilon)$  such that

$$L(\varepsilon) < f_n(x) - f_0 < R(\varepsilon).$$

Indeed, taking  $x_i = 0$  in the identity, both sides collapse to the value  $f_0$ . Continuity follows.  $\square$

*Proof 2* Apply instead Theorem 61.A of [Hal, Ch. XII, p. 266] to establish the base case, and then proceed inductively as before.  $\square$

**Corollary** *Theorem K holds for non-meagre Baire sets  $E$  in place of Borel sets in the form:*



for each  $p$  in  $\mathbb{N}$  there exists  $\varepsilon = \varepsilon_p(E) > 0$  such that

$(E + x_1) \cap \cdots \cap (E + x_p)$  is non-meagre, for  $|x_i| < \varepsilon$  ( $i = 1, \dots, p$ ).

*Proof* A non-meagre Baire set differs from an open set by a meagre set.  $\square$

We will now prove Theorem 5.1 using the Generic Completeness Principle (Theorem 3.4); this amounts to proceeding in two steps. To motivate the proof strategy, note that the embedding property is upward-hereditary (i.e. monotonic in the sense of Section 3): if  $T$  includes a subsequence of  $z_n$  by a shift  $t$  in  $T$ , then so does any superset of  $T$ . We first consider a non-meagre  $\mathcal{G}_\delta$ /non-null closed set  $T$ , just as in [BHW], modified to admit the consecutive format. We next deduce the theorem by appeal to  $\mathcal{G}_\delta$  inner-regularity of category/measure and Generic Dichotomy. (The subset  $E$  of exceptional shifts can only be meagre/null.)

**Theorem 5.2 (Generalized BHW Lemma—Existence of sequence embedding;** cf. [BHW, Lemma 2.2]) *For  $T$  Baire non-meagre/measurable non-null and a null sequence  $z_n \rightarrow 0$ , there exist  $t \in T$  and an infinite  $\mathbb{M}_t$  such that*

$$\{t + \bar{z}_{pm} : m \in \mathbb{M}_t\} \subseteq T.$$

*Proof* The conclusion of the theorem is inherited by supersets (is upward hereditary), so without loss of generality we may assume that  $T$  is Baire non-meagre/measurable non-null and completely metrizable, say under a metric  $\rho = \rho_T$ . (For  $T$  measurable non-null we may pass down to a compact non-null subset, and for  $T$  Baire non-meagre we simply take away a meagre set to leave a Baire non-meagre  $\mathcal{G}_\delta$  subset.) Since this is an equivalent metric, for each  $a \in T$  and  $\varepsilon > 0$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $B_\delta(a) \subseteq B_\varepsilon^\rho(a)$ . Thus, by taking  $\varepsilon = 2^{-n-1}$  the  $\delta$ -ball  $B_\delta(a)$  has  $\rho$ -diameter less than  $2^{-n}$ .

Working inductively in steps of length  $p$ , we define subsets of  $T$  (of possible translators)  $B_{pm+i}$  of  $\rho$ -diameter less than  $2^{-m}$  for  $i = 1, \dots, p$  as follows. With  $m = 0$ , we take  $B_0 = T$ . Given  $n = pm$  and  $B_n$  open in  $T$ , choose  $N$  such that  $|z_k| < \min\{\frac{1}{2}|x_n|, \varepsilon_p(B_n)\}$ , for all  $k > N$ . For  $i = 1, \dots, p$ , let  $x_{n-1+i} = z_{N+i} \in Z$ ; then by Theorem K or its Corollary  $B_n \cap (B_n - x_n) \cap \cdots \cap (B_n - x_{n+p})$  is non-empty (and open). We may now choose a non-empty subset  $B_{n+i}$  of  $T$  which is open in  $A$  with  $\rho$ -diameter less than  $2^{-m-1}$  such that  $\text{cl}_T B_{n+i} \subset B_n \cap (B_n - x_n) \cap \cdots \cap (B_n - x_{n+i}) \subseteq$

$B_{n+i-1}$ . By completeness, the intersection  $\bigcap_{n \in \mathbb{N}} B_n$  is non-empty. Let

$$t \in \bigcap_{n \in \mathbb{N}} B_n \subset T.$$

Now  $t + x_n \in B_n \subset T$ , as  $t \in B_{n+1}$ , for each  $n$ . Hence  $\mathbb{M}_t := \{m : z_{mp+1} = x_n \text{ for some } n \in \mathbb{N}\}$  is infinite. Moreover, if  $z_{pm+1} = x_n$  then  $z_{pm+2} = x_{n+1}, \dots, z_{pm+p} = x_{n+p-1}$  and so

$$\{t + \bar{z}_{pm} : m \in \mathbb{M}_t\} \subseteq T. \quad \square$$

We now apply Theorem 3.3 (Generic Dichotomy) to extend Theorem 5.2 from an existence to a genericity statement, thus completing the proof of Theorem 5.1.

**Theorem 5.3 (Genericity of sequence embedding)** *For  $T$  Baire/measurable and  $z_n \rightarrow 0$ , for generically all  $t \in T$  there exists an infinite  $\mathbb{M}_t$  such that*

$$\{t + \bar{z}_{pm} : m \in \mathbb{M}_t\} \subseteq T.$$

*Hence, if  $Z \subseteq X$  accumulates at 0 (has an accumulation point there), then for some  $t \in T$  the set  $Z \cap (T - t)$  accumulates at 0 (along  $Z$ ). Such a  $t$  may be found in any open set with which  $T$  has non-null intersection.*

*Proof* Working as usual in  $X$ , the correspondence

$$F(T) := \bigcap_{n \in \omega} \bigcup_{m > n} [(T - z_{pm+1}) \cap \dots \cap (T - z_{pm+p})]$$

takes Baire sets to Baire sets and is monotonic. Here  $t \in F(T)$  iff there exists an infinite  $\mathbb{M}_t$  such that  $\{t + \bar{z}_{pm} : m \in \mathbb{M}_t\} \subseteq T$ . By Theorem 5.2  $F(T) \cap T \neq \emptyset$ , for  $T$  Baire non-meagre, so we may appeal to Generic Dichotomy (Th. 3.3) to deduce that  $F(T) \cap T$  is quasi all of  $T$  (cf. Example 1 of Section 3).

With the main assertion proved, let  $Z \subseteq X$  accumulate at 0 and suppose that  $z_n$  in  $Z$  converges to 0. Take  $p = 1$ . Then, for some  $t \in T$ , there is an infinite  $\mathbb{M}_t$  such that  $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$ . Thus  $\{z_m : m \in \mathbb{M}_t\} \subseteq Z \cap (T - t)$  has 0 as a joint accumulation point.  $\square$

The preceding argument identifies only that  $Z \cap (T - t)$  has a point of simple, rather than essential, contiguity. More in fact is true, as we show in Theorem 5.4 below.

**Notation** Omitting the superscript if context allows, denote by  $\mathcal{M}_0^{Ba}$  resp.  $\mathcal{M}_0^{Leb}$  the family of Baire/Lebesgue-measurable sets which accumulate essentially at 0.

The following is a strengthened version of the two results in Lemma 2.4 (a) and (b) of [BHW] (embraced by (iii) below).

**Theorem 5.4 (Shifted-filter property of  $\mathcal{M}_0$ )** *Let  $A$  be Baire non-meagre/measurable non-null,  $B \in \mathcal{M}_0^{Ba/Leb}$ .*

- (i) *If  $(A - t) \cap B$  accumulates (simply) at 0, then  $(A - t) \cap B \in \mathcal{M}_0$ .*
- (ii) *For  $A, B \in \mathcal{M}_0$ , and generically all  $t \in A$ , the set  $(A - t) \cap B \in \mathcal{M}_0$ .*
- (iii) *For  $B \in \mathcal{M}_0$  and  $t$  such that  $(B - t) \cap B$  accumulates (simply) at 0, the set  $(B - t) \cap B$  accumulates essentially at 0.*

*Proof* We will prove (i) separately for the two cases (a) Baire (b) measure. From KBD (i) implies (ii), while (i) specializes to (iii) by taking  $A = B$ .

- (a) Baire case. Assume that  $A$  is Baire non-meagre and that  $B$  accumulates essentially at 0.

Suppose that  $A \cup N_1 = U \setminus N_0$  with  $U$  open, non-empty, and  $N_0$  and  $N_1$  meagre. Put  $M = N_0 \cup N_1$  and fix  $t \in A \setminus M$ , so that  $t$  is quasi-any point in  $A$ ; put  $M_t^- := M \cup (M - t)$ , which is meagre. As  $U \setminus M \subset A$ , note that by translation  $(U - t) \setminus (M - t) \subset A - t$ .

Let  $\varepsilon > 0$ . Without loss of generality  $B_\varepsilon(0) \subset U - t$ . By the assumption on  $B$ ,  $B \cap B_\varepsilon(0)$  is non-meagre, and thus so is  $[B \cap B_\varepsilon(0)] \setminus M_t^-$ . But the latter set is included in  $B \cap (A - t)$ ; indeed

$$[B \cap B_\varepsilon(0)] \setminus M_t^- \subset [B \cap (u - t)] \setminus (M - t) = B_\varepsilon(0) \cap B \cap (A - t).$$

As  $\varepsilon$  was arbitrary,  $B \cap (A - t)$  accumulates essentially at 0.

- (b) Measure case. Let  $A, B$  be non-null Borel, with  $B$  accumulating essentially at  $e$ . Without loss of generality both are density-open (all points are density points). By KBD,  $(A - t) \cap B$  accumulates (simply) at  $e$  for almost all  $t \in A$ . Fix such a  $t$ .

Let  $\varepsilon > 0$  be given. Pick  $x \in (A - t) \cap B$  with  $|x| < \varepsilon/2$  (possible since  $B \cap (A - t)$  accumulates at  $e$ ). As  $x$  and  $x - t$  are density points of  $B$  and  $A$  (resp.) pick  $\delta < \varepsilon/2$  such that

$$|B \cap B_\delta(x)| > \frac{3}{4}|B_\delta(x)| = \frac{3}{4}|B_\delta(e)|$$

and

$$|A \cap B_\delta(x - t)| > \frac{3}{4}|B_\delta(x - t)| = \frac{3}{4}|B_\delta(e)|,$$

which is equivalent to

$$|(A - t) \cap B_\delta(x)| > \frac{3}{4}|B_\delta(e)|.$$

By inclusion-exclusion as before, with  $A_1 := (A - t) \cap B_\delta(x)$  and  $A_2 := B \cap B_\delta(x)$ ,

$$|(A - t) \cap B \cap B_\delta(x)| > \frac{3}{2}|B_\delta(e)| - |B_\delta(e)| > 0.$$

But  $|x| < \varepsilon/2 < \varepsilon - \delta$  so  $|x| + \delta < \varepsilon$ , and thus  $B_\delta(x) \subseteq B_\varepsilon(e)$ , hence

$$|(A - t) \cap B \cap B_\varepsilon(e)| > 0.$$

As  $\varepsilon > 0$  was arbitrary,  $(A - t) \cap B$  is measurably large at  $e$ .  $\square$

## 6 Applications: additive combinatorics

Recall van der Waerden's theorem [vdW] of 1927, that in any finite colouring of the natural numbers, one colour contains arbitrarily long arithmetic progressions. This is one of Khinchin's three pearls of number theory [Kh, Ch. 1]. It has had enormous impact, for instance in Ramsey theory ([Ram1]; [GRS], [HS, Ch. 18]) and additive combinatorics ([TV]; [HS, Ch. 14]).

An earlier theorem of the same type, but for finite partitions of the *reals* into measurable cells, is immediately implied by the theorem of Ruziewicz [Ruz] in 1925, quoted below. We deduce its category and measure forms from the consecutive form of the KBD Theorem. The Baire case is new.

**Theorem R (Ruziewicz's Theorem** [Ruz]; cf. [Kem] after Lemma 2.1 for the measure case) *Given  $p$  positive real numbers  $k_1, \dots, k_p$  and any Baire non-meagre/measurable non-null set  $T$ , there exist  $d$  and points  $x_0 < x_1 < \dots < x_p$  in  $T$  such that*

$$x_i - x_{i-1} = k_i d, \quad i = 1, \dots, p.$$

*Proof* Given  $k_1, \dots, k_p$ , define a null sequence by the condition  $z_{pm+i} = (k_1 + \dots + k_i)2^{-m}$  ( $i = 1, \dots, p$ ). Then there are  $t \in T$  and  $m$  such that

$$\{t + z_{mp+1}, \dots, t + z_{mp+p}\} \subseteq T.$$

Taking  $d = 2^{-m}$ ,  $x_0 = t$  and for  $i = 1, \dots, p$

$$x_i = t + z_{mp+i} = t + (k_1 + \dots + k_i)d,$$

we have  $x_0 < x_1 < \dots < x_p$  and

$$x_{i+1} - x_i = k_i d. \quad \square$$

- Remarks** 1. If each  $k_i = 1$  above, then the sequence  $x_0, \dots, x_p$  is an arithmetic progression of arbitrarily small step  $d$  (which we can take as  $2^{-m}$  with  $m$  arbitrarily large) and arbitrarily large length  $p$ . So if  $\mathbb{R}$  is partitioned into a finite number of Baire/measurable cells, one cell  $T$  is necessarily non-meagre/non-null, and contains arbitrarily long arithmetic progressions of arbitrarily short step. This is similar to the van der Waerden theorem.
2. By referring to the continuity properties of the functions  $f_i$  in Theorem K, Kemperman strengthens the Ruziewicz result in the measure case, by establishing the existence of an upper bound for  $d$ , which depends on  $p$  and  $T$  only.

We now use almost completeness and the shifted-filter property (Th. 5.2) to prove the following.

**Theorem BHW** [BHW, Th. 2.6 and 2.7] *For a Baire/measurable set  $A$  which accumulates essentially at 0, there exists in  $A$  a sequence of reals  $\{t_n\}$  such that  $\sigma_F(t) := \sum_{i \in F} t_i \in A$  for every  $F \subseteq \omega$ .*

*Proof* As in Theorem 5.2 the conclusion is upward hereditary, so without loss of generality we may assume that  $A$  is completely metrizable (for  $A$  measurable non-null we may pass down to a compact non-null subset accumulating essentially at 0, and for  $A$  Baire non-meagre we simply take away a meagre set to leave a Baire non-null  $\mathcal{G}_\delta$  subset). Let  $\rho = \rho_A$  be a complete metric equivalent to the Euclidean metric. Denote by  $\rho$ -diam the  $\rho$  diameter of a set.

Referring to the shifted-filter property of  $\mathcal{M}_e^{Ba}$  or  $\mathcal{M}_e^{Leb}$ , we inductively choose decreasing sets  $A_n \subseteq A$  and points  $t_n \in A_n$ . Assume inductively that:

- (i)  $(A_n - t_n)$  accumulates at 0,
- (ii)  $\sigma_F = \sum_{i \in F} t_i \in A_{\max F}$ , for any finite set of indices  $F \subseteq \{0, 1, \dots, n\}$ ,
- (iii)  $\rho$ -diam( $\sigma_F$ )  $\leq 2^{-n}$  for all finite  $F \subseteq \{0, 1, \dots, n\}$ .

By Theorem 5.4,

$$A_{n+1} := A_n \cap (A_n - t_n) \text{ accumulates essentially at 0.}$$

Let  $\delta_n \in (0, t_n)$  be arbitrary (to be chosen later). By above, we may pick

$$t_{n+1} \in A_{n+1} \cap (0, \delta_n/2) \text{ such that } (A_{n+1} - t_{n+1}) \text{ accumulates at 0.}$$

Thus  $t_n$  is chosen inductively with  $t_{n+1} \in A_{n+1} \cap (A_{n+1} - t_{n+1})$  and  $\sum_{i \in I} t_i$  convergent for any  $I$ . Also

$$\sum_{i=n+1}^{\infty} t_i \leq t_{n+1} \sum_{i=n+1}^{\infty} 2^{-i} = \delta_n 2^{-n} < \delta_n.$$

Evidently  $t_1 \in A_1$ . As  $A_n \subset A_{n+1} \subset A_n - t_n$ , we see that, as  $t_1 + \cdots + t_n \in A_n$ , we have  $t_1 + \cdots + t_{n+1} \in A_{n+1}$ . More generally,  $\sigma_F = \sum_{i \in F} t_i \in A_{\max F}$  for any finite set of indices  $F \subseteq \{0, 1, \dots, n+1\}$ . For  $\varepsilon = 2^{-n-1}$  there exists  $\delta = \delta(\varepsilon) > 0$  small enough such that for all finite  $F \subseteq \{0, 1, \dots, n+1\}$

$$B_\delta(\sigma_F) \subseteq B_\varepsilon^\rho(\sigma_F).$$

Taking  $\delta_n < \delta(2^{-n-1})$  in the inductive step above implies that, for any infinite set  $I$ , the sequence  $\sigma_{I \cap \{0, \dots, n\}}$  is Cauchy under  $\rho$ , and so  $\sigma_I \in A$ .  $\square$

**Remark (Generalizations)** Much of the material here (which extends immediately from additive to multiplicative formats) can be taken over to the more general contexts of  $\mathbb{R}^d$  and beyond—to normed groups (including Banach spaces), for which see [BOst6]. We choose to restrict here to the line—Kingman’s setting—for simplicity, and in view of Mark Kac’s dictum: No theory can be better than its best example.

*Postscript* It is no surprise that putting a really good theorem and a really good mathematician together may lead to far-reaching consequences. We hope that John Kingman will enjoy seeing his early work on category still influential forty-five years later. The link with combinatorics is much more recent, and still pleases and surprises us—as we hope it will him, and our readers.

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